

# Tighter Bounds for Makespan Minimization on Unrelated Machines

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**Abstract.** We consider the problem of scheduling  $n$  jobs to minimize the makespan on  $m$  unrelated machines, where job  $j$  requires time  $p_{ij}$  if processed on machine  $i$ . A classic algorithm of Lenstra et al. [5] yields the best known approximation ratio of 2 for the problem. Improving this bound has been a prominent open problem for over two decades.

In this paper we obtain a tighter bound for a wide subclass of instances which can be identified efficiently. Specifically, we define the *feasibility factor* of a given instance as the minimum fraction of machines on which each job can be processed. We show that there is a polynomial-time algorithm that, given values  $L$  and  $T$ , and an instance having a *sufficiently large* feasibility factor  $h \in (0, 1]$ , either proves that no schedule of mean machine completion time  $L$  and makespan  $T$  exists, or else finds a schedule of makespan at most  $T + L/h$  which is smaller than  $2T$  for a wide class of instances.

For the restricted version of the problem, where for each job  $j$  and machine  $i$ ,  $p_{ij} \in \{p_j, \infty\}$ , we show that a simpler algorithm yields a better bound, thus improving for highly feasible instances the best known ratio of  $33/17 + \epsilon$ , for any fixed  $\epsilon > 0$ , due to Svensson [8].

## 1 Introduction

In the problem of scheduling on unrelated parallel machines, we are given a set  $\mathcal{J}$  of jobs to be processed without interruption on a set  $\mathcal{M}$  of unrelated machines, where the time a machine  $i \in \mathcal{M}$  needs to process a job  $j \in \mathcal{J}$  is specified by a machine and job dependent processing time  $p_{ij} \geq 0$ . When considering a scheduling problem, the most common and perhaps most natural objective function is makespan minimization. This is the problem of finding a schedule (or, an assignment),  $\alpha : \mathcal{J} \rightarrow \mathcal{M}$ , so as to minimize the time  $\max_{i \in \mathcal{M}} \sum_{j \in \alpha^{-1}(i)} p_{ij}$  required to process all jobs. A classic result in scheduling theory is the Lenstra-Shmoys-Tardos 2-approximation algorithm for this fundamental problem [5]. Their approach is based on several nice structural properties of the extreme point solutions of a natural linear program and has become a textbook example of such techniques (see, e.g., [10]). Complementing their positive result, they also proved that the problem is NP-hard to approximate within a factor less than  $3/2$ , even in the restricted case (also known as the *restricted assignment problem*), where  $p_{ij} \in \{p_j, \infty\}$ . Despite being a prominent open problem in scheduling theory,

there has been very little progress on either the upper or lower bound since the publication of [5] over two decades ago, with the exception being the recent beautiful result of Svensson [8], showing for the restricted case an upper bound of  $33/17 + \epsilon$ , for an arbitrarily small constant  $\epsilon > 0$ .

In this paper we show that these best known bounds can be tightened for a wide subclass of instances, which can be identified efficiently. Specifically, we define the *feasibility factor* of a given instance as the minimum fraction of machines on which each job can be processed. We show that there is a polynomial-time algorithm that, given values  $L$  and  $T$ , and an instance having a *sufficiently large* feasibility factor  $h \in (0, 1]$ , either proves that no schedule of mean machine completion time  $L$  and makespan  $T$  exists, or else finds a schedule of makespan at most  $T + L/h < 2T$ . For the restricted assignment problem, we show that a simpler algorithm yields a better bound, thus enabling to improve for highly feasible instances the best known ratio of  $33/17 + \epsilon$  of [8].

We note that the feasibility factor  $h$  of a given instance has been used before, both for improving upper bounds (see, e.g., [1]) and for showing the hardness of certain subclasses of instances [11]. However, these previous studies focus on specific values of  $h$  (see below). Our study takes a different approach in exploring the decrease in the makespan that can be achieved, by identifying instances in which  $h$  is sufficiently large (see Section 1.2).

## 1.1 Prior Work

Minimizing the makespan on unrelated parallel machines has been extensively studied for almost four decades. Lenstra et al. [5] introduced an LP-based polynomial time 2-approximation algorithm for the problem. They also proved that unless  $P = NP$ , there is no polynomial time approximation algorithm with approximation factor better than  $\frac{3}{2}$ . Gairing et al. [3] presented a combinatorial 2-approximation algorithm for the problem. Shchepin and Vakhania [6] showed that the rounding technique used in [5] can be modified to derive an improved ratio a factor of  $2 - \frac{1}{m}$ .

Shmoys and Tardos [7] further developed the technique of [5] to obtain an approximation ratio of 2 for the *generalized assignment problem (GAP)*, defined as follows. We are given a set of jobs,  $\mathcal{J}$ , and a set of unrelated machines,  $\mathcal{M}$ . Each job is to be processed by exactly one machine; processing job  $j$  on machine  $i$  requires time  $p_{ij} > 0$  and incurs a cost of  $c_{ij} > 0$ . Each machine  $i$  is available for  $T_i$  time units, and the objective is to minimize the total cost incurred. The paper [7] presents a polynomial time algorithm that, given values  $C$  and  $T$ , finds a schedule of cost at most  $C$  and makespan at most  $2T$ , if a schedule of cost  $C$  and makespan  $T$  exists. This is the best known result to date for GAP.

For the restricted assignment problem, Gairing et al. [2] presented a combinatorial  $2 - \frac{1}{p_{max}}$ -approximation algorithm based on flow techniques, where  $p_{max} = \max_j p_j$  is the maximum processing time of any job in the given instance. The best known approximation ratio is  $\frac{33}{17} + \epsilon = 1.9412 + \epsilon$ , due to Svensson [8].

Interestingly, the feasibility factor of a given instance served as a key component in deriving two fundamental results for the restricted assignment problem. Ebenlendr et al. [1] showed that the subclass of instances for which  $h = 2/m$  admits an approximation factor of 1.75, thus improving for this subclass the general bound of 2. The same problem, also called *unrelated graph balancing*, was studied by Verschae and Wiese [11]. They showed that, in fact, this surprisingly simple subclass of instances constitutes the core difficulty for the *linear programming* formulation of the problem, often used as a first step in obtaining approximate solutions. Specifically, they showed that already for this basic setting, the strongest known LP-formulation, namely, the configuration-LP, has an integrality gap of 2.

## 1.2 Our Contribution

In this paper we improve the best known bounds for makespan minimization on unrelated parallel machine, for a wide subclass of instances possessing high feasibility factor. In particular, in Section 3, we show that there is a polynomial-time algorithm that, given values  $0 < L < T$ , and an instance  $I$  having a feasibility factor  $L/T \leq h$ , either proves that no schedule of mean machine completion time  $L$  and makespan  $T$  exists, or else finds a schedule of makespan at most  $T + L/h < 2T$ .

For the restricted assignment problem, let  $L = \frac{\sum_{j=1}^n p_j}{m}$  be the mean machine completion time of *any* schedule. Then we show that there is an  $O(m^2n)$  time algorithm that, given an instance  $I$  whose feasibility factor satisfies  $L/p_{max} = q < h$ , finds a schedule of makespan at most  $p_{max} + L/h < (1 + q/h)OPT$ , where  $OPT$  is the makespan of an optimal schedule. Thus, for  $q \leq \frac{16}{17}h$ , this improves the bound of  $33/17 + \epsilon$  of [8].

**Techniques:** Our algorithms rely heavily on the fact that the given instances are highly feasible, and thus, the schedules can be better balanced to decrease the latest completion time of any job. Our algorithm for general instances first uses as a subroutine an algorithm of [7], thus also identifying highly feasible instances. It then applies on the resulting schedule a *balancing* phase. Using some nice properties of this schedule, the algorithm moves long jobs from overloaded to underloaded machines, while decreasing the makespan of the schedule.

Our bound for the restricted assignment problem builds on a result of Gairing et al. [2], who gave an algorithm based on flow techniques for general instances of the problem. Their algorithm starts with a schedule satisfying certain properties and gradually improves the makespan until it is guaranteed to yield a ratio of 2 to the optimal. The main idea is to use some parameters for partitioning the machines into three sets: overloaded, underloaded, and all the remaining machines. The makespan is improved by moving jobs from overloaded to underloaded machines on augmenting path in the corresponding flow network. We adopt this approach and show that by a *good* selection of the parameters defining the three machine sets, we can obtain the desired makespan. Consequently, our algorithm is simpler and has better running time than the algorithm of [2] (see Section 4).

## 2 Preliminaries

An *assignment* of jobs to machines is given by a function  $\alpha : \mathcal{J} \rightarrow \mathcal{M}$ . Thus,  $\alpha(j) = i$  if job  $j$  is assigned to machine  $i$ . For any assignment  $\alpha$ , the *load*  $\delta_i$ , on machine  $i$ , given a matrix of processing times  $\mathbf{P}$ , is the sum of processing times for the jobs that were assigned to machine  $i$ , thus  $\delta_i(\mathbf{P}, \alpha) = \sum_{j \in \mathcal{J} : \alpha(j)=i} p_{ij}$ . The *makespan* of an assignment  $\alpha$  is the maximum load on any machine. Also, the *average machine load* (or, mean machine completion time) is given by  $L = \frac{\sum_{i \in \mathcal{M}} \delta_i(\mathbf{P}, \alpha)}{m}$ .

Given the matrix  $\mathbf{P}$  and the value  $T > 0$ , we say that a machine  $i$  is *legal* for job  $j$  if  $p_{ij} \leq T$ . Thus, any job  $j \in \mathcal{J}$  can be assigned to at least  $hm$  machines in  $\mathcal{M}$ . The *feasibility factor* of  $\mathbf{P}$  is  $h(T) = \frac{\min_{j \in \mathcal{J}} |\{i \in \mathcal{M} : i \text{ is legal for } j\}|}{m}$ .

Given an assignment  $\alpha$  for a matrix  $\mathbf{P}$ , and a constant  $\gamma \geq 1$ , we denote by  $Bad(\mathbf{P}, \alpha, \gamma)$  the set of machines that complete processing after time  $T + \gamma \cdot L$ , by  $Good(\mathbf{P}, \alpha, \gamma)$  the set of machines that complete by time  $\gamma \cdot L$ , and by  $Good_j(\mathbf{P}, \alpha, \gamma)$  the set of machines from  $Good(\mathbf{P}, \alpha, \gamma)$  that are legal for job  $j$ . Also we denote by  $j_{max}^i(\mathbf{P}, \alpha) = \operatorname{argmax} \{p_{ij} : \alpha(j) = i\}$  the largest job on machine  $i \in Bad(\mathbf{P}, \alpha, \gamma)$ . We denote by  $G_{\alpha, \gamma}(\mathbf{P}, \alpha)$  the bipartite graph  $((Bad(\mathbf{P}, \alpha, \gamma), Good(\mathbf{P}, \alpha, \gamma)), E(\mathbf{P}, \alpha))$  where  $E(\mathbf{P}, \alpha)$  consists of all edges  $(i, i')$  where  $i'$  is a legal machine for  $j_{max}^i(\mathbf{P}, \alpha)$ . We say that machine  $i$  is *good* if  $i \in Good(\mathbf{P}, \alpha, \gamma)$ . We say that machine  $i$  is *bad* if  $i \in Bad(\mathbf{P}, \alpha, \gamma)$ . Machine  $i$  is *good for job*  $j$  if  $i \in Good_j(\mathbf{P}, \alpha, \gamma)$ . We omit  $\mathbf{P}$  in the notation if it is clear from the context.

## 3 Approximation Algorithm for General Instances

The problem of scheduling on unrelated machines can be viewed as a special case of the *generalized assignment problem* in which  $c_{ij} = 0$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . The best known result for the generalized assignment problem is due to [7]. They presented a polynomial time algorithm that, given values  $C$  and  $T$ , finds a schedule of cost at most  $C$  and makespan at most  $2T$ , if a schedule of cost  $C$  and makespan  $T$  exists. This implies also the best result for scheduling on unrelated machines - a schedule of makespan at most twice the optimum. In our result we use the rounding technique of [7] with the costs being the processing times. This allows us to bound the average machine's completion time of the resulting assignment. A bound that is essential for our result. In the next Section we give an overview of the rounding technique of [7].

### 3.1 Overview of the Algorithm of Shmoys and Tardos

We describe below the technique used in [7] for solving the generalized assignment problem. Let  $\mathbf{P}$  and  $\mathbf{C}$  denote the matrix of processing times and the matrix of costs, and let  $T$  and  $C$  be fixed positive integers. Let the indicator variables  $x_{ij}$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$  denote whether job  $j$  is assigned to machine  $i$ . Then the linear programming relaxation of the problem is as follows:

$$\begin{aligned}
LP(\mathbf{P}, \mathbf{C}, T, C) : \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \leq C \\
& \sum_{i=1}^m x_{ij} = 1, \quad \text{for } j = 1, \dots, n \\
& \sum_{j=1}^n p_{ij} x_{ij} \leq T, \quad \text{for } i = 1, \dots, m \\
& x_{ij} \geq 0, \quad \text{for } i = 1, \dots, m, j = 1, \dots, n \\
& x_{ij} = 0, \text{ if } p_{ij} > T, \text{ for } i = 1, \dots, m, j = 1, \dots, n
\end{aligned}$$

Let  $x_{ij}$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$  be the fractional solution, and let  $k_i = \lceil \sum_{j=1}^n x_{ij} \rceil$ . Each machine is partitioned into  $k_i$  sub-machines  $v_{i,s}$ ,  $s = 1, \dots, k_i$ . The rounding is done by finding a minimum-cost perfect matching between all jobs and all sub-machines. Formally, a bipartite graph  $B(W, V, E)$  is constructed, with  $W = \{w_j : j = 1, \dots, n\}$  the job nodes,  $V = \{v_{i,s} : i = 1, \dots, m, s = 1, \dots, k_i\}$  the sub-machines nodes. Each node  $v_{i,s}$  can be viewed as a bin of volume 1, and we add an edge  $(w_j, v_{i,s})$  with cost  $c_{ij}$  iff a positive fraction of  $x_{ij}$  is packed in the bin  $v_{i,s}$ . For every machine  $i = 1, \dots, m$  the jobs are sorted in non-increasing order of their processing time on  $i$ . Then, the bins  $v_{i1}, \dots, v_{ik_i}$  are packed one by one, with the values  $x_{ij} > 0$  by the order of the jobs. While  $v_{i,s}$  is not totally packed, we continue packing the  $x_{i,j}$ s such that if  $x_{ij}$  fits  $v_{i,s}$  it is packed to  $v_{i,s}$ , else, only a fraction  $\beta$  of  $x_{ij}$  is packed to  $v_{i,s}$ , consuming all the remaining volume of  $v_{i,s}$ ; the remaining part of  $(1 - \beta)x_{ij}$  is packed in  $v_{i,s+1}$ . Then the rounding is done by taking a minimum-cost integer matching  $M$ , that matches all job nodes, and for each edge  $(w_j, v_{i,s}) \in M$ , set  $x_{ij} = 1$ , i.e., schedule job  $j$  on machine  $i$ .

The resulting schedule has the following nice property, that is used below for deriving our result.

**Lemma 1.** [7] *Let  $\alpha$  be the assignment obtained by the algorithm and let  $j_{max}^i = \max \{p_{ij} : \alpha(j) = i\}$  the longest jobs that was assigned to  $i$ . Then, for all  $1 \leq i \leq m$ ,  $\sum_{j: \alpha(j)=i, j \neq j_{max}^i} p_{ij} \leq T$ .*

### 3.2 Approximation Algorithm

Consider the special case of the generalized assignment problem in which the costs satisfy  $c_{ij} = p_{ij}$ , for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . For any instance  $\mathbf{P}$  and the constants  $T$  and  $L \leq T$ , integral solutions to the linear program,  $LP(\mathbf{P}, \mathbf{P}, T, L)$ , are in one-to-one correspondence with schedules of makespan at most  $T$ , and average machine load  $L$ .

Thus, the result of [7] guarantees that if  $LP(\mathbf{P}, \mathbf{P}, T, L)$  has a feasible solution, then there exists a schedule that has makespan at most  $2T$  and average machine load  $L$ . Our main result is the following.

**Theorem 1.** *Let  $T$  and  $L \leq T$  be some fixed positive values for a given instance  $\mathbf{P}$  of the scheduling problem, let  $h(T) = h \in (0, 1]$  be the feasibility factor of  $\mathbf{P}$ . If  $LP(\mathbf{P}, T, L)$  has a feasible solution, then there is an algorithm that achieves a makespan of  $\min \{T + \frac{L}{h}, 2T\}$ .*

We prove the theorem by describing an algorithm that converts a feasible solution for  $LP(\mathbf{P}, \mathbf{P}, T, L)$  to the desired schedule. The first step of the algorithm is to apply the rounding technique of [7] to obtain an assignment  $\alpha$  that admits makespan at most  $2T$  and average machine load at most  $L$ . Next, the algorithm fixes the assignment to achieve a makespan of at most  $T + \frac{L}{h}$ . This is done by transferring the largest job from each machine  $i \in \text{Bad}(\mathbf{P}, \alpha, \frac{1}{h})$  to a machine  $i' \in \text{Good}(\mathbf{P}, \alpha, \frac{1}{h})$ . To prove that all transfers are possible, we show that there exists a perfect matching between the bad machines and the good machines. Formally, we prove that there exists a perfect matching in  $G_{\alpha, \frac{1}{h}}(\mathbf{P}, \alpha)$ . We first prove the following lemmas.

**Lemma 2.** *Let  $\alpha$  be an assignment that admits a makespan of at most  $2T$  and average machine load  $L \leq T$ . Let  $\gamma \geq 1$  and assume  $|\text{Bad}(\alpha, \gamma)| = k$ . Then*

- (i)  $k < \frac{m}{\gamma+1}$ .
- (ii)  $|\text{Good}(\alpha, \gamma)| > \left(1 - \frac{1}{\gamma}\right) \cdot m + \frac{k}{\gamma} \cdot \frac{T}{L}$ .

*Proof.* Each machine  $i \in \text{Bad}(\alpha, \gamma)$  has load greater than  $T + \gamma \cdot L$ , therefore  $\sum_{i \in M} \delta_i(\alpha) > k \cdot (T + \gamma \cdot L)$ .

- (i) Assume that  $k \geq \frac{m}{\gamma+1}$ , then

$$\begin{aligned} \sum_{i \in M} \delta_i(\alpha) &> k(T + \gamma L) \\ &\geq \frac{m}{\gamma+1}(T + \gamma L) \\ &= \frac{m}{\gamma+1}(T + \gamma L) \\ &\geq \frac{m}{\gamma+1}(L + \gamma L) \\ &= m \cdot L \end{aligned}$$

The last inequality follows from the fact that  $T \geq L$ . Hence, the average machine load is greater than  $L$ , a contradiction. It follows that  $k < \frac{m}{\gamma+1}$ .

- (ii) Let  $|\text{Good}(\alpha, \gamma)| = l$ . Then, there are  $m - k - l$  machines having loads greater than  $\gamma L$ . Assume that  $l \leq \left(1 - \frac{1}{\gamma}\right) m + \frac{k}{\gamma} \cdot \frac{T}{L}$ , then

$$\begin{aligned} \sum_{i \in M} \delta_i(\alpha) &> k(T + \gamma \cdot L) + (m - l - k) \gamma L \\ &= kT + (m - l) \gamma L \\ &\geq kT + \left(m - \left(1 - \frac{1}{\gamma}\right) m + \frac{k}{\gamma} \cdot \frac{T}{L}\right) \gamma L \\ &\geq kT + \left(\frac{m}{\gamma} + \frac{k}{\gamma} \cdot \frac{T}{L}\right) \gamma L \\ &= kT + (mL + kT) \\ &\geq mL \end{aligned}$$

Hence, the average machine load is greater than  $L$ , a contradiction. It follows that  $|\text{Good}(\alpha, \gamma)| \geq \left(1 - \frac{1}{\gamma}\right) \cdot m + \frac{k}{\gamma} \cdot \frac{T}{L}$ .  $\square$

**Lemma 3.** *Let  $\mathbf{P}$  be an instance of the scheduling problem. Let  $\alpha$  be an assignment for  $\mathbf{P}$  that admits a makespan of at most  $2T$  and average machine load  $L \leq T$ . If  $h \geq \frac{L}{T}$  then for every subset  $A \subseteq \text{Bad}(\alpha, \frac{1}{h})$ ,  $|N(A)| \geq |A|$ , where  $N(A)$  is the set of neighbors of  $A$  in  $G_{\alpha, \gamma}$ .*

*Proof.* Let  $|\text{Bad}(\alpha, \frac{1}{h})| = k$ . Since the number of illegal machines for any job  $j$  is at most  $(1-h)m$ , the number of good machines for job  $j$  is at least the number of good machines minus its illegal machines (the worst case where all illegal machines for job  $j$  form a subset of  $\text{Good}(\alpha, \frac{1}{h})$ ). Together with Lemma 2 we have

$$\begin{aligned} |\text{Good}_j(\alpha, \frac{1}{h})| &\geq |\text{Good}(\alpha, \frac{1}{h})| - (1-h)m \\ &> \left(1 - \frac{1}{\gamma}\right) \cdot m + \left(\frac{k}{h}\right) \cdot \frac{T}{L} - (1-h)m \\ &= h \cdot k \frac{T}{L} \\ &\geq k \end{aligned}$$

The last inequality follows from the fact that  $h \geq \frac{L}{T}$ . Now, let  $A \subseteq \text{Bad}(\alpha, \frac{1}{h})$ . Then  $|A| \leq |\text{Bad}(\alpha, \frac{1}{h})| = k$ . Recall that the set of neighbors of  $A$  is the set of machines that are good for all the jobs  $j_{max}^i$ ,  $i \in A$ , i.e.,  $N(A) = \cup_{i \in A} \text{Good}_{j_{max}^i}(\alpha, \frac{1}{h}) \subseteq \text{Good}(\alpha, \frac{1}{h})$ . Obviously  $|N(A)| = |\cup_{i \in A} \text{Good}_{j_{max}^i}(\alpha, \frac{1}{h})| \geq |\text{Good}_{j_{max}^i}(\alpha, \frac{1}{h})|$  for some  $i \in A$ . It follows from the above that  $|N(A)| \geq k$ .

Since  $|A| \leq k$  we have that  $|N(A)| \geq |A|$ .  $\square$

By Hall's Theorem [4], there exist a perfect matching in  $G_{\alpha, \frac{1}{h}}$  iff for every  $A \subseteq \text{Bad}(\alpha, \frac{1}{h})$ ,  $|N(A)| \geq |A|$ . Thus, we have

**Corollary 1.** *There exists a perfect matching in  $G_{\alpha, \frac{1}{h}}$ .*

By the above discussion, we can modify the assignment  $\alpha$ , output by the algorithm of [7], by finding a perfect matching in  $G_{\alpha, \frac{1}{h}}$  and then transferring jobs from bad machines to their matching good machines. We describe this formally in algorithm  $A_{UM}(\mathbf{P}, \gamma, T, L)$ .

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**Algorithm 1**  $A_{UM}(\mathbf{P}, \gamma, T, L)$

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- (i) Solve the linear relaxation  $LP(\mathbf{P}, T, L)$ .
  - (ii) Round the solution to obtain an integral assignment  $\alpha$  using a rounding technique as given in [7].
  - (iii) If the feasibility factor  $h$  of  $\mathbf{P}$  satisfies  $h \leq \frac{L}{T}$ , then return the assignment  $\alpha$ .
  - (iv) Otherwise, construct the bipartite graph  $G_{\alpha, \frac{1}{h}}$  and find a perfect matching of size  $|\text{Bad}(\mathbf{P}, \alpha)|$ .
  - (v) Obtain a resulting assignment  $\beta$  out of  $\alpha$  by transferring the longest job,  $j_{max}^i$  from each machine  $i \in \text{Bad}(\mathbf{P}, \alpha)$ , to its matching machine  $i' \in \text{Good}(\mathbf{P}, \alpha)$ .
  - (vi) Return the new assignment  $\beta$ .
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**Proof of Theorem 1.** We show that the assignment output by Algorithm  $A_{UM}$  satisfies the statement of the theorem. Consider an instance  $\mathbf{P}$ . By [7], if  $LP(\mathbf{P}, \mathbf{P}, T, L)$  has a feasible solution for  $C \leq T$  then Step 2. is guaranteed to generate a schedule of makespan at most  $2T$  and average machine load  $L$ . Let  $\alpha$  be the resulting assignment.

Let  $h$  be the feasibility factor of  $\mathbf{P}$ . If  $h \leq \frac{C}{T}$  then we output  $\alpha$  at Step 3, and indeed, we cannot guarantee a makespan lower than  $2T$  in this case. Otherwise, by Corollary 1, there exists a perfect matching in  $G_{\alpha, \frac{1}{h}}$ . Let  $|Bad(\mathbf{P}, \alpha)| = k$  and let  $M = \{(i_{b_1}, i_{g_1}), \dots, (i_{b_k}, i_{g_k})\}$  be a perfect matching in  $G_{\alpha, \frac{1}{h}}$ .

By Lemma 1, for any machine  $i = 1, \dots, m$  the sum of processing times of all the jobs  $j$  such that  $j \in \{j : \alpha(j) = i\} \setminus \{j_{max}^i\}$  is at most  $T$ . Therefore, transferring the largest job  $j_{max}^i$  to another machine guarantees that the new load of  $i$  is at most  $T$ .

As for the good machines, if  $i$  is a good machine for job  $j$  then  $p_{ij} \leq T$ , then transferring  $j$  to  $i$  will increase the load of  $i$  at most by  $T$ . Since the load of a good machine is at most  $\frac{L}{h}$ , we have that after such job transfer the load is at most  $T + \frac{L}{h}$ .

We note that each pair  $(i_{b_s}, i_{g_s}) \in M$  is a matching between  $i_{b_s} \in Bad$  and  $i_{g_s} \in Good$ .  $i_{b_s}$  and therefore transferring the largest job  $j_{max}^{i_{b_s}}$  from  $i_{b_s}$  to  $i_{g_s}$  guarantees that the resulting load on  $i_{b_s}$  is at most  $T$ , and the load on  $i_{g_s}$  is at most  $T + \frac{L}{h}$ .

Thus, by performing the large-jobs transfers for all pairs  $(i_{b_s}, i_{g_s}) \in M$ ,  $s = 1, \dots, k$ , we guarantee that each machine has load at most  $T + \frac{L}{h}$ .  $\square$

## 4 A Better Bound for the Restricted Assignment Problem

In this section we consider the restricted version of our problem, where  $p_{ij} \in \{p_j, \infty\}$ , for each job  $j = 1, \dots, n$ , and each machine  $i = 1, \dots, m$ . For this variant, we show that improved approximation ratio can be achieved by a combinatorial algorithm. In particular, applying a technique of Gairing et al. [2], we show that, by identifying highly feasible instances, we obtain an algorithm which improves the 2-approximation ratio guaranteed in [2], and also has better running time. In the next section we give an overview of the algorithm of [2].

### 4.1 Overview of the Algorithm of Gairing et al.

We describe below an algorithm, called Unsplittable-Blocking-Flow, introduced in [2]. Let  $I$  be an instance for the restricted assignment problem. Also, let  $w$  be a fixed positive integer and  $\Delta$  a parameter (to be determined). A  $w$ -feasible assignment  $\alpha$  is an assignment with the property that each job  $j$  is assigned to a machine  $i$  where  $p_{ij} \leq w$ .

Let  $\alpha$  be a  $w$ -feasible assignment.  $G_\alpha(w) = (W \cup V, E_\alpha(w))$  is a directed bipartite graph where  $W = \{w_j : j = 1, \dots, n\}$  consists of the job nodes, and



$V = \{v_i : i = 1, \dots, m\}$  consists of machine nodes. For any job node  $j$  and any machine node  $i$ , if  $\alpha(j) = i$  there is an arc in  $E_\alpha(w)$  from  $i$  to  $j$ ; if  $\alpha(j) \neq i$  and  $j$  is feasible on machine  $i$ , i.e.,  $p_{ij} \leq w$ , then there is an arc from  $j$  to  $i$ .

Given a  $w$ -feasible assignment  $\alpha$ , the algorithm of [2] partitions the set of machines to three subsets:  $\mathcal{M}^+$  (overloaded),  $\mathcal{M}^-$  (underloaded), and  $\mathcal{M}^0$  (all the remaining machines). Thus,  $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \cup \mathcal{M}^0$ . Given an assignment  $\alpha$ , a machine  $i \in \mathcal{M}^+$  is overloaded if the load on  $i$  is at least  $w + \Delta + 1$ . A machine  $i \in \mathcal{M}^-$  is underloaded if the load on  $i$  is at most  $\Delta$ . The remaining machines, which are neither overloaded nor underloaded, form the set  $\mathcal{M}^0 = \mathcal{M} \setminus (\mathcal{M}^- \cup \mathcal{M}^+)$ .

The algorithm Unsplittable-Blocking-Flow( $\mathcal{M}, \alpha, \Delta, w$ ) starts with an initial  $w$ -feasible assignment of jobs to machines and iteratively improves the makespan until it obtains an assignment with makespan of  $w + \Delta$ , or declares that an assignment of makespan  $\Delta$  does not exist. In each iteration, the algorithm finds an augmenting path from an overloaded to an underloaded machine, and pushes jobs along this path, by performing a series of job reassignments between machines on that path. This results in balancing the load over the machines, i.e., reducing the load of the source that is an overloaded machine, and increasing the load of the destination that is an underloaded machine, while preserving the load of all other machines. Unsplittable-Blocking-Flow terminates after  $O(mS)$  steps, where  $S = \sum_{j=1}^n |\{i : p_{ij} < \infty\}|$ . For short, we call this algorithm below  $\mathcal{UBF}$ .

Algorithm  $\mathcal{UBF}$  (with  $w = p_{max}$ ), combined with a binary search over the possible range of values for  $\Delta$ , can be used to obtain the approximation ratio of  $2 - \frac{1}{p_{max}}$ . The running time of the approximation algorithm is then factored by a value that is logarithmic in the size of the range in which we search for  $\Delta$ . Thus, the algorithm of [2] computes an assignment having makespan within a factor of  $2 - \frac{1}{p_{max}}$  from the optimal in time  $O(mS \log W)$ , where  $W = \sum_{j=1}^n p_j$ .

## 4.2 Approximation Algorithm

Let  $\mathcal{I}$  be an instance of the restricted assignment problem. The feasibility factor of  $\mathcal{I}$  is exactly  $h = \frac{\min_j |\{i : p_{ij} < \infty\}|}{m}$ . Consider the bipartite graph  $G_\alpha(w)$  constructed by algorithm  $\mathcal{UBF}$ .

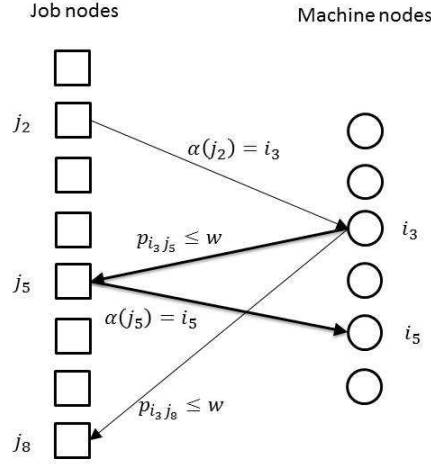
Let  $L = \frac{1}{m} \sum_{j=1}^n p_j$  be the average machine load of any schedule for an instance of the restricted assignment problem. In the following we define the three machine sets for the algorithms.

**Definition 1.** Let  $\alpha$  be a  $w$ -feasible assignment of a given instance  $\mathcal{I}$  of the restricted assignment problem, with feasibility factor  $h$ . Then,

$$\begin{aligned}\mathcal{M}^-(\alpha) &= \{i : \delta_i(\mathcal{I}, \alpha) \leq \frac{L}{h}\} \\ \mathcal{M}^0(\alpha) &= \{i : \frac{L}{h} < \delta_i(\mathcal{I}, \alpha) \leq w + \frac{L}{h}\} \\ \mathcal{M}^+(\alpha) &= \{i : \delta_i(\mathcal{I}, \alpha) > w + \frac{L}{h}\}\end{aligned}$$

Using this partition of machines into  $\mathcal{M}^0$ ,  $\mathcal{M}^-$  and  $\mathcal{M}^+$ , we apply the algorithm  $\mathcal{UBF}(\mathcal{M}, \alpha, \frac{L}{h}, p_{max})$  of [2].

Throughout the execution of  $\mathcal{UBF}$ , augmenting paths from machines in  $\mathcal{M}^+$  to  $\mathcal{M}^-$  are found iteratively. Along each of these paths, the algorithm reassigns jobs between machines. Applying the algorithm results in reducing the makespan and balancing the loads. The algorithm continues as long as there exists a path from  $\mathcal{M}^+$  to  $\mathcal{M}^-$ .



**Fig. 1.** The bipartite graph  $G_\alpha(w)$ . By changing the orientation of the path  $i_3 \rightarrow j_5 \rightarrow i_5$ , we remove job  $j_5$  from  $i_5$  and schedule it on machine  $i_3$ .

**Theorem 2.** Let  $\mathcal{I}$  be an instance of the restricted scheduling problem, with feasibility factor  $h \in (0, 1]$ . If there exists  $0 < q < h$  such that  $\frac{\sum_{j=1}^n p_j}{m} = q \cdot p_{\max}$  then there exists a  $(1 + \frac{q}{h})$ -approximation algorithm for the makespan, whose running time is  $O(m^2 n)$ .

The correctness of Theorem 2 follows from the next lemma.

**Lemma 4.** Let  $\mathcal{I}$  be an instance of the restricted scheduling problem, with feasibility factor  $h \in (0, 1]$ . Then Algorithm  $\mathcal{UBF}$  takes time  $O(mS)$ , where  $S = \sum_{j=1}^n |\{i : p_{ij} < \infty\}|$ . Furthermore, For an initial assignment  $\alpha$ ,  $\mathcal{UBF}(\mathcal{M}, \alpha, \frac{L}{h}, w)$  for  $w \geq p_{\max}$  and  $L = \frac{1}{m} \sum_{j=1}^n p_j$  terminates with  $\mathcal{M}^+ = \phi$ .

*Proof.* Let  $w \geq p_{max}$ . We show that  $\mathcal{UBF}(\mathcal{M}, \alpha, \frac{L}{h}, w)$  terminates with  $M^+ = \phi$ . Let  $\beta$  be the assignment computed by  $\mathcal{UBF}(\mathcal{M}, \alpha, \frac{L}{h}, w)$ . By the observation, the average machine load of  $\beta$  is  $L$ . Assume that  $\mathcal{M}^+ \neq \emptyset$ . Then there exists a machine  $v$  with load  $\delta_v(\mathcal{I}, \beta) > w + \frac{L}{h}$ . Since  $\mathcal{UBF}$  terminated, we know that there is no path from a machine in  $\mathcal{M}^+$  to a machine in  $\mathcal{M}^-$  in the graph  $G_\beta(w)$ . Denote by  $\mathcal{M}_v$  the set of machines reachable from  $v$ ,  $\mathcal{M}_v = \{i \in M : \text{there is a directed path in } G_\beta(w) \text{ from } v \text{ to } i\}$ .

Obviously, there is at least one job  $u$  which is assigned to  $v$  in  $\beta$ . Thus, there is an edge of  $(v, u)$  in  $G_\beta(w)$ . If  $h$  is the feasibility factor of  $\mathcal{I}$ , then there exists at least  $hm$  machines that are good for  $u$ , which means that there exists an edge from  $u$  to each of them. By appending each of these edges to  $(v, u)$  we get a directed path from  $v$  to at least  $hm$  machines (including  $v$ ). Therefore we can conclude that  $|\mathcal{M}_v| \geq hm$ .

We compute now a lower bound of the average machine load for  $\beta$ . We sum the loads of all the machines  $i \in \mathcal{M}$ . We know that  $v \in \mathcal{M}^+$ , thus  $\delta_v(\mathcal{I}, \beta) > w + \frac{L}{h}$ . We also know that there is no path from  $v$  to machines in  $\mathcal{M}^-$ , and therefore  $\mathcal{M}_v \cap \mathcal{M}^- = \emptyset$ . Thus, for all  $i \in \mathcal{M}_v$   $\delta_i(\mathcal{I}, \beta) > \frac{L}{h}$  holds.

$$\begin{aligned} \sum_{i \in \mathcal{M}} \delta_i(\mathcal{I}, \beta) &\geq \delta_v(\mathcal{I}, \beta) + \sum_{i \in \mathcal{M}_v, i \neq v} \delta_i(\mathcal{I}, \beta) \\ &> w + \frac{L}{h} + (|\mathcal{M}_v| - 1) \frac{L}{h} \\ &= w + |\mathcal{M}_v| \left(\frac{L}{h}\right) \\ &\geq w + mh \left(\frac{L}{h}\right) \\ &= w + mL \\ &> mL \end{aligned}$$

We have shown that the sum of loads of the assignment  $\beta$  is greater than  $mL$ . Hence, the average load for  $\beta$  is greater than  $L$  in contradiction to the average load of the resulting assignment  $\beta$  being exactly  $L$ . For the analysis of the running time, see [2].  $\square$

**Proof of Theorem 2.** Let  $\mathcal{I}$  be an instance of the restricted scheduling problem with feasibility factor  $h$ . Let  $w = p_{max}$  and let  $\alpha$  be some initial  $w$ -feasible assignment. The average machine load of  $\alpha$  is  $L = \frac{1}{m} \sum_{j=1}^n p_j$ . By Lemma 4, when  $\mathcal{UBF}(\mathcal{M}, \alpha, \frac{L}{h}, w)$  terminates, we have that  $\mathcal{M}^+ = \emptyset$ , i.e., all machines are in  $\mathcal{M}^0$  or in  $\mathcal{M}^-$ . Therefore, the maximum load of the resulting assignment is at most  $w + \frac{L}{h} = p_{max} + \frac{L}{h} = p_{max} \left(1 + \frac{L}{hp_{max}}\right)$ . Since the optimal makespan satisfies  $OPT \geq p_{max}$ , and  $q = \frac{L}{p_{max}}$ , we have an approximation ratio of  $1 + \frac{q}{h} < 2$ . By Lemma 4, the running time of our algorithm is  $O(mS) = O(m^2n)$ .  $\square$

Note that our algorithm has better running time than the algorithm of [2], which uses binary search to find the best value for  $\Delta$  yielding a 2-approximation ratio to the minimum makespan. This results in an overall running time of  $O(mS \log W)$ , where  $W = \sum_{j=1}^n p_j$  is the sum of processing times of all jobs.

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